Math 142 Lecture 2 Notes

Daniel Raban

January 18, 2018

1 Limit Points, Closure, and Continuity

1.1 Limit points and closure

Definition 1.1. Let $A \subseteq X$ and $p \in X$. Then p is a *limit point* of A if every neighborhood U of p satisfies $U \cap (A \setminus \{p\}) \neq \varnothing$; i.e. U has a point of A besides p.

Remark 1.1. A limit point of a set may not be contained in the set.

Theorem 1.1. *$A \subseteq X$ is closed iff A contains all its limit points.*

Proof. ($\Longrightarrow$) If X is closed, then $X \setminus A$ is open. So for any $x \in X \setminus A$, $X \setminus A$ us a neighborhood of x. But $(X \setminus A) \cap (A \setminus \{x\}) = \varnothing$. So x is not a limit point of A. So any limit point of A is in A.

($\Longleftarrow$) If A contains all its limit points, we want to show that $X \setminus A$ is open. If $x \in X \setminus A$, it is not a limit point, so there exists a neighborhood U_x of x with $U_x \cap (A \setminus \{x\}) = \varnothing$. So $U_x \subseteq X \setminus A$. Then $X \setminus A = \bigcup_{x \in X \setminus A} U_x$ is a union of open sets making it open. So A is closed. $\square$

Definition 1.2. If $A \subseteq X$, the *closure* of A is

$$\overline{A} := A \cup \{\text{limit points of A}\}.$$

Theorem 1.2. *$\overline{A}$ is the smallest closed set containing A.*

Proof. If $A \subseteq B \subseteq X$ and B is closed, any limit point of A is a limit point of B. B is closed, so B contains all its limit points; then B contains all the limit points of A. So $\overline{A} \subseteq B$.

We need to show that $\overline{A}$ is closed. Let $x \in X \setminus \overline{A}$; then x is not a limit point of A. So there exists a neighborhood U_x of x such that $U_x \subseteq X \setminus A$. We want to show that $U_x \subseteq X \setminus \bar{A}$. If $y \in U_x$ is a limit point of A, then U_x is a neighborhood of y, and $U_x \cap A = \varnothing$. But y is a limit point, so such a neighborhood shouldn't exist. So $U_x \cap \overline{A} = \varnothing$; i.e. $U_x \subseteq X \setminus A$. So $X \setminus \overline{A} = \bigcup_{x \in X \setminus \overline{A}} U_x$, making it open. So $\bar{A}$ is closed. $\square$

Corollary 1.1. *$A \subseteq X$ is closed iff $A = \overline{A}$.*

Definition 1.3. A *base* of a topological space X is a collection of open sets such that if $A \subseteq X$ is open, A is a union of open sets in the collection.

Example 1.1. $\mathbb{R}^n$ with the usual topology has base $\{B_\varepsilon(x) : x \in \mathbb{R}^n, \varepsilon > 0\}$.[1]

1.2 Continuity

Definition 1.4. A function $f : X \to Y$ is *continuous* if $f^{-1}(A) \subseteq X$ is open whenever $A \subseteq Y$ is open.[2]

A continuous function is often called a *map*.

Theorem 1.3. *If $A \subseteq X$ has the subspace topology, then the inclusion $i : A \to X$, sending $a \mapsto a$, is continuous.*

Proof. If $U \subseteq X$ is open, then

$$i^{-1}(U) = \{a \in A : i(a) \in U\} = A \cap U,$$

which is open by the definition of the subspace topology. □

Theorem 1.4. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.*

Proof. Note that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. If $A \subseteq Z$ is open, then $g^{-1}(A)$ is open, as g is continuous. So $f^{-1}(g^{-1}(A))$ is open, as f is continuous. This says that $(g \circ f)^{-1}(A)$ is open, so $g \circ f$ is continuous. □

Corollary 1.2. *If $f : X \to Y$ is continuous and $A \subseteq X$ has the subspace topology, then $f|_A : A \to Y$ is continuous, where $f|_A(a) = f(i(a))$.*

Theorem 1.5. *The following are equivalent.*

1. *$f : X \to Y$ is continuous.*
2. *$f^{-1}(A)$ is closed whenever $A \subseteq Y$ is closed.*
3. *If $\{U_\alpha\}$ is a base for the topology on Y, then $f^{-1}(U_\alpha)$ is open for all α.*

Proof. See textbook. □

Example 1.2. Let X be a set with the discrete topology, let Y be any set with any topology, and let $f : X \to Y$ be any function. Then f is continuous, as $f^{-1}(A) \subseteq X$ is always open for any subset $A \subseteq Y$.

[1] Last lecture, we used this notation to mean closed balls. Here, we mean open, so we are using the "$<$" notation, rather than "$\leq$."

[2] While f might not have an inverse, we mean here that $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

Example 1.3. Continuity from anaylsis is the same as continuity in topology, when they both apply. If $f : (X, d_x) \to (Y, d_y)$ is a function of metric spaces, then f is "analysis continuous" if for all $x \in X$ and $\varepsilon > 0$, $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. So if $A \subseteq Y$ is open, we want to show that $f^{-1}(A)$ is open. So if $x \in f^{-1}(A)$, then $f(x) \in A$. So if A open, there exists some $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq A$. Since f is "analysis continuous," there exists a $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq A$. So $B_\delta(x) \subseteq f^{-1}(A)$, and then $f^{-1}(A)$ is open. So if f is "analysis continuous," f is "topology continuous." The converse is left as an exercise.